Macro-Spatial Economics

Lecture 3: Extensions and Special Cases (Allen-Arkolakis II)

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Recap from Last Week

Simple quantitative model:

- Geography: locations with innate productivity $\bar{A}_i$, amenities $\bar{u}_i$, trade costs τ_{ij}
- Production: Armington, price $p_{ij} = \tau_{ij} w_i / A_i$
- Consumption: CES preferences, welfare $W_i = w_i u_i / P_i$
- Agglomeration: $A_i = \bar{A}_i L_i^{\alpha}$, $u_i = \bar{u}_i L_i^{\beta}$

Today: Dive deeper into equilibrium structure and special cases

Defining Equilibrium

Geography: $\{\tau_{ij}, \bar{A}_i, \bar{u}_i\}_{i,j\in\mathcal{N}}$

Model parameters: $\{\sigma, \alpha, \beta\}$

Distribution of economic activity: $\{L_i, w_i, P_i\}_{i\in\mathcal{N}}, W$

1. **Goods market clearing**:

$$w_i L_i = \sum_{j\in\mathcal{N}} X_{ij} \quad \forall i \in \mathcal{N}$$

2. **Budget constraints are satisfied** :

$$w_i L_i = \sum_{j\in\mathcal{N}} X_{ji} \quad \forall i \in \mathcal{N}$$

3. **Welfare is equalized**: There exists a scalar $W > 0$, s.t.

$$W_i \leq W \quad \forall i \in \mathcal{N}, \text{ with equality if } L_i > 0$$

4. **Population constraint**:

$$\sum_{i\in\mathcal{N}} L_i = \bar{L}$$

4.2.1: The Equilibrium System

$$X_{ij} = \tau_{ij}^{1-\sigma} \left(\frac{w_i}{A_i} \right)^{1-\sigma} P_j^{\sigma-1} w_j L_j \tag{21}$$

$$W_i = \frac{w_i u_i}{P_i} \tag{18}$$

Combining equilibrium conditions yields two key equations:

Equilibrium Conditions

$$W^{\sigma-1} w_i^{\sigma} L_i^{1-\alpha(\sigma-1)} = \sum_{j \in \mathcal{N}} \left(\bar{A}_i \bar{u}_j / \tau_{ij} \right)^{\sigma-1} w_j^{\sigma} L_j^{1+\beta(\sigma-1)} \tag{31}$$

$$W^{\sigma-1} w_i^{1-\sigma} L_i^{\beta(1-\sigma)} = \sum_{j \in \mathcal{N}} \left(\bar{u}_i \bar{A}_j / \tau_{ji} \right)^{\sigma-1} w_j^{1-\sigma} L_j^{\alpha(\sigma-1)} \tag{32}$$

These are $2N$ equations in $2N + 1$ unknowns: $\{w_i, L_i\}_{i=1}^{N}$ and W

Plus constraint: $\sum_i L_i = \bar{L}$

Understanding the System

Structure of equilibrium system (31)–(32):

- Both involve sums over all locations
- Wages and populations in location i depend on all other locations
- Total productivities $A_i = \bar{A}_i L_i^{\alpha}$ and amenities $u_i = \bar{u}_i L_i^{\beta}$ create non-linearities

Key insight: System can be simplified dramatically using market access

Next

The next step rewrites these in terms of market access and derives intuitive labor supply/demand curves

Reformulating with Market Access

Goal: Express bilateral trade flows more intuitively

Start with gravity equation:

$$X_{ij} = \tau_{ij}^{1-\sigma} \left(\frac{w_i}{A_i}\right)^{1-\sigma} P_j^{\sigma-1} w_j L_j$$

Substitute into goods market clearing $w_i L_i = \sum_j X_{ij}$:

$$\left(\frac{w_i}{A_i}\right)^{1-\sigma} = \frac{w_i L_i}{\sum_{j\in\mathcal{N}} \tau_{ij}^{1-\sigma} P_j^{\sigma-1} w_j L_j} \tag{33}$$

Define **outward market access**:

$$\Pi_i^{1-\sigma} \equiv \sum_{j\in\mathcal{N}} \tau_{ij}^{1-\sigma} P_j^{\sigma-1} w_j L_j$$

Gravity with Market Access

Substituting back into gravity equation:

$$X_{ij} = \tau_{ij}^{1-\sigma} \times \frac{w_i L_i}{\Pi_i^{1-\sigma}} \times \frac{w_j L_j}{P_j^{1-\sigma}} \tag{34}$$

Interpretation:

- Trade decreases with bilateral costs $\tau_{ij}^{1-\sigma}$
- Origin's share: $w_i L_i / \Pi_i^{1-\sigma}$ (income divided by outward MA)
- Destination's share: $w_j L_j / P_j^{1-\sigma}$ (income divided by inward MA)

Key property: Market access terms summarize how location interacts with rest of world

Deriving Labor Demand

Substitute agglomeration $A_i = \bar{A}_i L_i^{\alpha}$ into equation (33):

$$(w_i/A_i)^{1-\sigma} = w_i L_i / \Pi_i^{1-\sigma}$$

Rearranging:

$$w_i^{-\sigma} L_i^{-\alpha(\sigma-1)} \bar{A}_i^{-(\sigma-1)} = w_i L_i / \Pi_i^{1-\sigma}$$

Take logs and solve for $\ln w_i$:

$$\ln w_i = -\left(\frac{1}{\sigma} - \alpha\frac{\sigma-1}{\sigma}\right) \ln L_i + \frac{1}{\sigma} \ln \Pi_i^{1-\sigma} + \frac{\sigma-1}{\sigma} \ln \bar{A}_i$$

This is the (inverse) labor demand curve!

Labor Demand: Economic Interpretation

$$\ln w_i = -\underbrace{\left(\frac{1}{\sigma} - \alpha\frac{\sigma-1}{\sigma}\right)}_{\text{local elasticity}} \ln L_i + \underbrace{\frac{1}{\sigma}}_{\text{global elasticity}} \ln \Pi_i^{1-\sigma} + \frac{\sigma-1}{\sigma} \ln \bar{A}_i$$

Slope: Downward if $\alpha < 1/(\sigma - 1)$

- Standard case: more workers $\Rightarrow$ lower wages (diminishing returns)
- Agglomeration attenuates the slope

Shifters:

- Better outward market access $\Pi_i \Rightarrow$ higher labor demand
- Higher innate productivity $\bar{A}_i \Rightarrow$ higher labor demand

Deriving Labor Supply

From welfare equalization $W_i = w_i u_i / P_i = W$:

$$w_i = WP_i / u_i$$

Substitute congestion $u_i = \bar{u}_i L_i^{\beta}$:

$$w_i = WP_i / (\bar{u}_i L_i^{\beta})$$

Take logs:

$$\ln w_i = -\beta \ln L_i + \frac{1}{1-\sigma} \ln P_i^{1-\sigma} + \ln W - \ln \bar{u}_i$$

This is the (inverse) labor supply curve!

Labor Supply: Economic Interpretation

$$\ln w_i = \underbrace{-\beta}_{\text{local elasticity}} \ln L_i + \underbrace{\frac{1}{1-\sigma}}_{\text{global elasticity}} \ln P_i^{1-\sigma} + \ln W - \ln \bar{u}_i$$

Slope: Upward if $\beta < 0$ (congestion)

- More workers $\Rightarrow$ worse amenities.
- Requires higher wages to attract more workers

Shifters:

- Better inward market access P_i $\Rightarrow$ lower cost of living $\Rightarrow$ labor supply shifts down (workers accept lower wage)
- Higher innate amenities $\bar{u}_i$ $\Rightarrow$ labor supply shifts down

The Role of Market Access

Critical insight: Market access summarizes all spatial linkages

If we could treat MA as exogenous:

- Simple equilibrium: equate supply and demand in each location separately
- Solve for $\{w_i, L_i\}$ given $\{P_i, \Pi_i\}$

But MA is endogenous!

- Π_i depends on $\{P_j, w_j, L_j\}$ in all locations j
- P_i depends on $\{w_j, L_j\}$ in all locations j
- Creates system of $2N$ equations

Key achievement

By modeling externalities, labor supply and demand now depend on LOCAL labor (not just MA), creating well-behaved equilibrium

Special Case (1): No Agglomeration or Congestion

Set $\alpha = \beta = 0$

Equilibrium conditions (31) and (32) simplify to:

$$W^{\sigma-1} w_i^{\sigma} L_i = \sum_{j \in \mathcal{N}} (\bar{A}_i \bar{u}_j / \tau_{ij})^{\sigma-1} w_j^{\sigma} L_j \tag{37}$$

$$W^{\sigma-1} w_i^{1-\sigma} = \sum_{j \in \mathcal{N}} (\bar{u}_i \bar{A}_j / \tau_{ji})^{\sigma-1} w_j^{1-\sigma} \tag{38}$$

Key property: Same log-linear combination on both sides!

- Left side: $w_i^{\sigma}, L_i, w_i^{1-\sigma}$
- Right side:$w_j^{\sigma}, L_j, w_j^{1-\sigma}$

Matrix Representation

Define:

- $\mathbf{x} = \{x_i\}$ where $x_i \equiv w_i^\sigma L_i$
- $\mathbf{y} = \{y_i\}$ where $y_i \equiv w_i^{1-\sigma}$
- $\mathbf{T}$ matrix with $T_{ij} = (\bar{A}_i \bar{u}_j / \tau_{ij})^{\sigma-1}$ = Geography
- $\lambda = W^{\sigma-1}$

Then equations become:

$$\lambda \mathbf{x} = \mathbf{T}\mathbf{x} \tag{39}$$

$$\lambda \mathbf{y} = \mathbf{T}'\mathbf{y} \tag{40}$$

This is an eigenvalue problem!

- $\mathbf{x}, \mathbf{y}$ are eigenvectors of $\mathbf{T}, \mathbf{T}'$
- λ is the eigenvalue

Perron-Frobenius Theorem

Result: For matrix $\mathbf{T}$ with positive entries:

- Unique positive eigenvalue λ (Perron-Frobenius root)
- Corresponding eigenvector is unique (up to scale) and positive
- This determines equilibrium distribution $\{L_i, w_i\}$

Interpretation:

- Workers attracted to productive locations ($\bar{A}_i$ high)
- Workers attracted to pleasant locations ($\bar{u}_i$ high)
- But also to locations *near* productive/pleasant places (through τ_{ij})

Welfare: $\lambda \propto W^{\sigma-1}$, so welfare proportional to eigenvalue

Social Savings

Without externalities ($\alpha = \beta = 0$), equilibrium is efficient

Welfare elasticity to trade cost reduction:

$$-\frac{\partial \ln W}{\partial \ln \tau_{ij}} = \frac{X_{ij}}{Y^W} \tag{42}$$

Interpretation:

- Welfare gains from reducing τ_{ij} proportional to trade share X_{ij}
- First-order effects only depend on initial trade flows
- Foundation of "social savings" approach (Fogel 1964)

Applications

Donaldson & Hornbeck (2016): railroads; Allen & Arkolakis (2014): highways

Special Case (2): Two Location Economy

Setup:

- $N = 2$
- Assume $A_1 > A_2$ (location 1 more productive)
- Symmetric trade costs: $\tau_{12} = \tau_{21} = \tau$
- No innate amenity differences
- No Externalities: $\alpha = \beta = 0$:

The general system of equations become a 2nd order equation in $\frac{w_1}{w_2}$:

$$\frac{A_2^{\sigma-1}}{A_1^{\sigma-1}} \left(\frac{w_1^{\sigma-1}}{w_2^{\sigma-1}} \right)^2 + \left(1 - \frac{A_2^{\sigma-1}}{A_1^{\sigma-1}} \right) \tau^{1-\sigma} \frac{w_1^{\sigma-1}}{w_2^{\sigma-1}} - 1 = 0$$

Question: How does population split between locations?

Method: Solve system for wage ratio w_1/w_2 and population shares

Two Locations: No Externalities

Key result: $\frac{w_1}{w_2} > 1$ (Intuitive)

Also: w_1/w_2 increases with τ (when $\sigma > 1$)

- Lower trade costs $\Rightarrow$ higher productivity more "shared"
- Higher trade costs $\Rightarrow$ tendency for worker to move in more productive region

Population: Location 1 gets more than half

- Share *increases* with trade costs τ
- Workers move to productive location when trade difficult

Contrast with Krugman (1991)

Krugman Core-Periphery: Population in dominant region *decreases* with trade costs

Allen-Arkolakis (no externalities): Population in dominant region *increases* with trade costs

Why the difference?

- Krugman: Agglomeration through firm entry only
- Market access effect dominates when trade costs fall
- AA: Agglomeration modeled differently (externalities)
- Creates different comparative statics

Special Case (3): Symmetric Trade Costs

Assumption: $\tau_{ij} = \tau_{ji}$ for all i, j

Key result: Inward and outward market access are proportional!

$$P_i^{1-\sigma} = \kappa \Pi_i^{1-\sigma}$$

for some scalar $\kappa > 0$

Proof sketch:

- Write market access equations (43) and (44)
- Guess $P_i = \kappa \Pi_i$ and substitute
- Show this satisfies both equations

Market Access Equations

Inward market access:

$$P_i^{1-\sigma} = \sum_{j \in \mathcal{N}} \tau_{ji}^{1-\sigma} \Pi_j^{\sigma-1} w_j L_j$$

Outward market access:

$$\Pi_i^{1-\sigma} = \sum_{j \in \mathcal{N}} \tau_{ij}^{1-\sigma} P_j^{\sigma-1} w_j L_j$$

If $\tau_{ij} = \tau_{ji}$ and $P_i = \kappa \Pi_i$, then both equations become:

$$\kappa \Pi_i^{1-\sigma} = \sum_{j \in \mathcal{N}} \tau_{ij}^{1-\sigma} \Pi_j^{\sigma-1} w_j L_j$$

Simplification with Symmetric Costs

Combine welfare equalization with proportionality $P_i = \kappa \Pi_i$:

$$L_i^{1+(\beta-\alpha)(\sigma-1)} \propto \frac{w_i^{1-2\sigma}}{(\bar{A}_i/\bar{u}_i)^{\sigma-1}}$$

Implication: Log-linear relationship between L_i and w_i!

Advantage: Reduces the $2N$ equlibrium equations (31) and (32) to N equations:

$$\frac{W}{\bar{L}^{\frac{\alpha+\beta}{\sigma-1}}} l_i^{\tilde{\sigma}\left(1-\alpha(\sigma-1)-\beta\sigma\right)} = \sum_{j\in\mathcal{N}} \tau_{ij}^{1-\sigma} \left(\bar{A}_i \bar{u}_j\right)^{(\sigma-1)\tilde{\sigma}} \left(\bar{u}_i \bar{A}_j\right)^{-\sigma\tilde{\sigma}} l_j^{\tilde{\sigma}\left(1+\sigma\alpha+\beta(\sigma-1)\right)}$$

where $\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}$

- Once we get l_i that solve the system, we get w_i.
- Significant computational savings
- Assuming agglomeration is not too strong, exists a unique solution.

When is Symmetry Reasonable?

Exact symmetry: $\tau_{ij} = \tau_{ji}$

- Natural if trade costs are only distance
- May not hold with borders, tariffs, regulations

Quasi-symmetry: $\tau_{ij} = \tau_{ji}\tau_i^A\tau_j^B$

- Allows origin-specific and destination-specific components
- Still permits simplification
- See Allen, Arkolakis & Takahashi (2020)

Applications

Many empirical papers impose symmetry for tractability when N is large (e.g., county-level analysis)

New Models, Same Old Equilibrium

Remarkable finding: Many different economic assumptions yield *mathematically identical* equilibrium!

Simple model assumptions:

- Perfect labor mobility
- Identical agents
- Perfect competition
- Armington trade structure
- Externalities only through α, β

Alternative microfoundations that yield same equilibrium:

- Ricardian comparative advantage
- Housing markets
- Heterogeneous preferences
- Imperfect mobility
- Roundabout production
- Multiple sectors

Why This Matters

Flexibility: Can choose microfoundation based on application

- Trade context: Ricardian
- Urban context: Housing
- Migration: Heterogeneous preferences

Tractability: All map to same labor supply/demand framework

- Different elasticities $\varepsilon^{S,D}_{\text{local}}$, $\varepsilon^{S,D}_{\text{global}}$
- Same solution methods
- Unified estimation approach

Robustness: Results not specific to particular assumptions

5.1: Ricardian Comparative Advantage

Setup: Perfect competition with Eaton-Kortum (2002) structure

- Continuum of goods, each supplied by the least-cost location
- Productivities drawn from a Fréchet distribution: $\varepsilon_i(\omega) \sim e^{-A_i z^{-\theta}}$

Result: Expenditure share of j on goods from i:

$$\lambda_{ij} = \frac{\left(\frac{w_i \tau_{ij}}{A_i}\right)^{-\theta}}{\sum_{i'} \left(\frac{w_{i'} \tau_{i'j}}{A_{i'}}\right)^{-\theta}}$$

This is isomorphic to gravity equation (21) with $\theta = \sigma - 1$

Key implication

Armington (love of variety) and Ricardian (least-cost supplier) microfoundations deliver the same aggregate trade elasticity — the two are observationally equivalent

5.2: Non-Tradable Sector

Motivation: Not all goods are tradable (services, housing)

Setup à la Helpman (1998):

- Workers spend fraction δ on tradable differentiated goods
- Fraction $1-\delta$ on a locally produced non-tradable good
- Non-tradable sector perfectly competitive; returns distributed equally across local workers

Effect: Only the labor *supply* curve changes

Isomorphism

Equivalent to baseline model with congestion externality on amenities:

$$\beta = -\frac{1-\delta}{\delta}$$

A larger non-tradable share $\Rightarrow$ stronger effective congestion

5.3: Fixed Factors of Production

Motivation: Labor is not the only input — land and capital also matter

Setup à la Donaldson & Hornbeck (2016):

- Production function with a fixed factor (e.g. capital K_i):

$$Y_i = A_i K_i^{\tilde{\alpha}} L_i^{1-\tilde{\alpha}}$$

- $\tilde{\alpha}$: share of the fixed factor

Effect: Fixed factor creates diminishing returns to labor

Isomorphism

Equivalent to baseline model with productivity congestion:

$$\tilde{\alpha} = -\alpha$$

A fixed factor acts as a **negative** productivity externality — it reduces the slope of labor demand, exactly as $\alpha < 0$ does

5.4: Endogenous Labor Supply

Motivation: Agents choose how many hours to work, not just where to live

Setup: Preferences over consumption *and* labor supply:

$$U(C_i, L_i) = C_i u_i - \frac{L_i^{1+\eta}}{1+\eta}$$

First-order conditions yield labor supply:

$$L_i = \left(\frac{w_i u_i}{P_i}\right)^{1/\eta}$$

Effect: Upward-sloping local labor supply; slope governed by η

Isomorphism

Equivalent to baseline model with amenity congestion:

$$-\beta = \eta$$

A higher elasticity of labor supply η acts like a stronger congestion externality on amenities

5.5: Idiosyncratic Preferences

Motivation: Workers are not identical — they have heterogeneous tastes over locations

Setup à la Redding (2016): welfare of worker ω in location j:

$$\frac{w_j}{P_j} u_j \times \varepsilon_j(\omega), \qquad \varepsilon_j(\omega) \sim \text{Fréchet}$$

Workers sort to the location that maximises utility. The resulting location shares are:

$$L_i/\bar{L} = \frac{(w_i u_i / P_i)^\theta}{\sum_j (w_j u_j / P_j)^\theta}$$

Effect: Location shares follow a gravity-type equation in real wages

Isomorphism

Equivalent to baseline labor supply equation (36) with:

$$\beta = -1/\theta$$

More heterogeneous preferences (lower θ) $\Rightarrow$ stronger effective congestion:

5.6: Idiosyncratic Productivities

Motivation: Workers differ in productivity across locations, not just preferences

Setup à la Bryan & Morten (2019): $\varepsilon_j(\omega)$ reinterpreted as idiosyncratic *productivity*

- Workers who sort into i tend to be more productive there
- Effective labor in location i:

$$\tilde{L}_i = L_i^{(\theta-1)/\theta}$$

- Modified labor market clearing: $\sum_i \tilde{L}_i^{\theta/(\theta-1)} = \bar{L}^{(\theta-1)/\theta}$

Isomorphism

Equivalent to baseline model with amenity congestion:

$$\beta = -\frac{1}{\theta - 1}$$

Heterogeneity in productivities, like heterogeneity in preferences, acts as a congestion externality on amenities

5.7: Monopolistic Competition with Free Entry

Motivation: The baseline model assumes perfect competition. Can we nest the Krugman (1991) structure?

Recall: Krugman (1991) features

- Monopolistic competition and increasing returns
- Free entry drives profits to zero
- Number of firms $N_i \propto L_i$ (from zero-profit condition)

Comparison: Labor supply & demand equations (14)–(15) of Krugman vs. (35)–(36) of the simple model

Isomorphism

Monopolistic competition with free entry $\equiv$ baseline model with:

$$\alpha = \frac{1}{\sigma - 1}$$

Free entry generates variety gains that act exactly like a productivity agglomeration force

Summary: What We Learned Today

Equilibrium structure

- System of $2N$ equations in wages and populations
- Goods market clearing + welfare equalization

Market access reformulation

- Gravity with inward/outward market access
- Labor supply and demand curves
- MA summarizes all spatial linkages

Three special cases

- No externalities: eigenvalue problem
- Two locations: role of trade costs
- Symmetric costs: major simplification

Alternative microfoundations

- Many models $\Rightarrow$ same equilibrium structure
- Flexibility + tractability + robustness

Next Week: The Workhorse Framework

Lecture 4: General framework and taking to data

- Workhorse quantitative framework
- Existence and uniqueness results
- Counterfactual analysis
- Estimation methods
- Applications and lessons learned

Reading:

- Allen & Arkolakis (2025) Sections 6-8
- Donaldson & Hornbeck (2016) - Railroads
- Ahlfeldt et al. (2015) - Berlin Wall