Allen–Arkolakis Italy Model
Hat Algebra and Estimation

Roberto Pancrazi

European University Institute & University of Warwick

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Hat Algebra: The Key Idea

Problem: to evaluate a counterfactual policy, the full model requires the unobserved fundamentals $\bar{A}_i$ and $\bar{u}_i$.

Solution: rewrite equilibrium in *changes* ("hats"): $\hat{x} \equiv x'/x$ where x' is counterfactual and x is baseline.

What hat algebra needs

- Baseline trade shares π_{ij} (observable from trade data)
- Baseline output $Y_i = w_i L_i$ and population L_i (observable)
- Elasticities σ, α, β (estimated or calibrated)
- The shock $\hat{K}_{ij}$ (the policy you want to evaluate)

What it does not need: $\bar{A}_i$, $\bar{u}_i$, or the level of trade costs τ_{ij}.
The baseline trade shares absorb all information about fundamentals.

The Hat Algebra System

Given exogenous shock $\hat{K}_{ij}$ (trade cost or productivity change), solve for $\{\hat{w}_i, \hat{L}_i\}$:

Equilibrium in changes

1. **Price index:** $\hat{P}_j^{1-\sigma} = \sum_i \pi_{ij}\, \hat{K}_{ij}\, \hat{w}_i^{1-\sigma}\, \hat{L}_i^{\alpha(\sigma-1)}$

   — uses baseline trade shares π_{ij} as weights

2. **New trade shares and market clearing:** $\hat{w}_i \hat{L}_i Y_i = \sum_j \pi_{ij}\, \hat{\pi}_{ij}\, \hat{w}_j \hat{L}_j Y_j$

   — revenue = sum of expenditure from all destinations

3. **Free mobility:** $\hat{w}_i\, \hat{u}_i\, \hat{L}_i^{\beta} \,/\, \hat{P}_i = \hat{W} \quad \forall\, i$

   — utility equalized; pins down $\hat{L}_i$ given $\hat{w}_i$

4. **Labour clearing:** $\sum_i L_i\, \hat{L}_i = \bar{L}$

Iterate on $(\hat{w}, \hat{L})$ with damping until convergence.

Encoding the Shock: $\hat{K}_{ij}$

The matrix $\hat{K}_{ij}$ is the only exogenous input. It encodes two types of policy:

Trade cost change
$\hat{K}_{ij} = \hat{\tau}_{ij}^{1-\sigma}$

Example: HSR corridor reduces τ by 30% between Milan and Naples.
$\hat{\tau}_{ij} < 1 \;\Rightarrow\; \hat{K}_{ij} > 1$
Lower trade costs increase the weight of that pair.

Productivity change
$\hat{K}_{ij} = \hat{\bar{A}}_i^{\sigma-1}$

Example: 20% Mezzogiorno subsidy raises $\bar{A}$ for Southern regions.
Applied to all j for a given origin i.
A more productive origin ships more to every destination.

In both cases, $\hat{K}_{ij} = 1$ for unaffected pairs (no change).

Exact vs. Linear Comparative Statics

Exact hat algebra

- Iterates on $(\hat{w}, \hat{L})$
- Fully nonlinear
- Exact for any shock size
- Same answer as full re-solve

Linear approximation

- Linearises around baseline
- Builds $2N \times 2N$ Jacobian
- Solves $J\,dx = -F_{\text{shock}}$ once
- Good for small shocks; drifts for large ones

Accuracy vs. shock size (Mezzogiorno subsidy)

Subsidy	Full solve	Exact hat	Linear
5%	✓	≈ exact	≈ exact
20%	✓	≈ exact	small drift
50%	✓	≈ exact	visible drift

The code sweeps the subsidy from 1% to 50% and plots the RMSE of each method (Figure 3).

Two Experiments

Experiment 1: HSR Corridor (Milan–Naples)

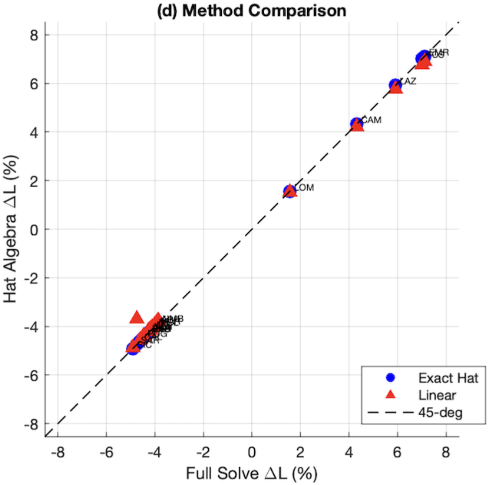
- 30% reduction in τ along consecutive pairs on the line (LOM–EMR–TOS–LAZ–CAM)
- 15% reduction for non-adjacent pairs (e.g. LOM–LAZ)
- All three methods compared region by region

Experiment 2: Mezzogiorno Subsidy

- 20% increase in $\bar{A}_i$ for all eight Southern regions
- Equivalent to a broad-based productivity programme
- Figures show maps of ΔL and scatter: full solve vs. hat algebra

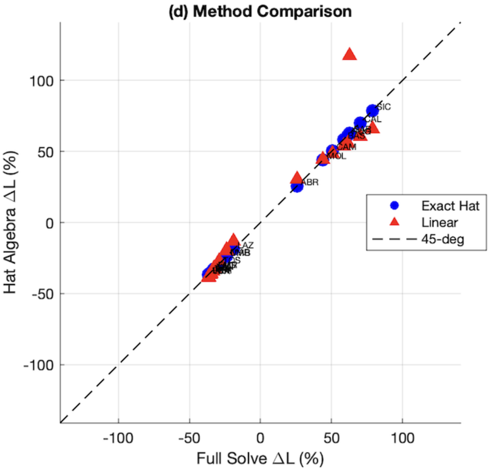
Takeaway: exact hat algebra reproduces the full model exactly. The linear approximation is a useful shortcut for small shocks.

HSR Corridor: Full Solve vs. Hat Algebra



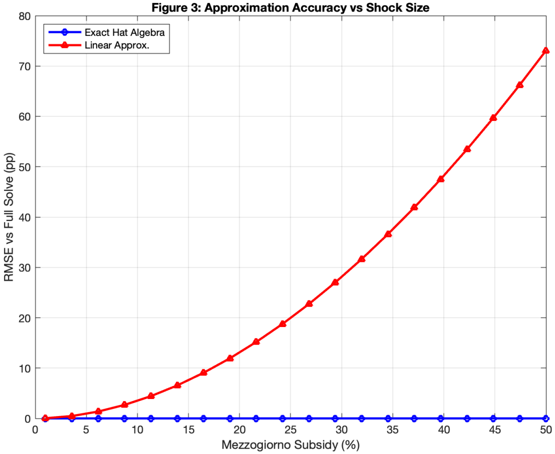
Panel (d): exact hat algebra lies on the 45-degree line; linear approximation is close but slightly off.

Mezzogiorno Subsidy: Full Solve vs. Hat Algebra



A 20% productivity subsidy to all Southern regions. Same pattern: exact hat ≈ full solve.

Approximation Accuracy vs. Shock Size



Exact hat algebra stays accurate at all shock sizes. The linear approximation drifts as the subsidy grows — that is the cost of dropping higher-order terms.

Bringing the Model to the Data

The calibrated model is a *laboratory*: we know the true parameters, so we can test whether estimation strategies recover them.

Six steps

1. **Gravity estimation** — recover ρ from bilateral trade flows (PPML)
2. **Network trade costs** — model τ_{ij} through a transport network
3. **Market access** — compute inward/outward MA from trade data
4. **Elasticity estimation** — recover α and β from data
5. **Market access approach** — approximate counterfactuals without re-solving
6. **Model testing** — check the model is correctly specified

Reference: Allen & Arkolakis (2025), Section 7.

Gravity Estimation (PPML)

The AA model implies a **gravity equation**:

$$X_{ij} = \underbrace{\tau_{ij}^{1-\sigma}}_{f(\text{distance})} \times \frac{Y_i}{\mathrm{MA}_i^{\mathrm{out}}} \times \frac{E_j}{\mathrm{MA}_j^{\mathrm{in}}}$$

Two estimators:

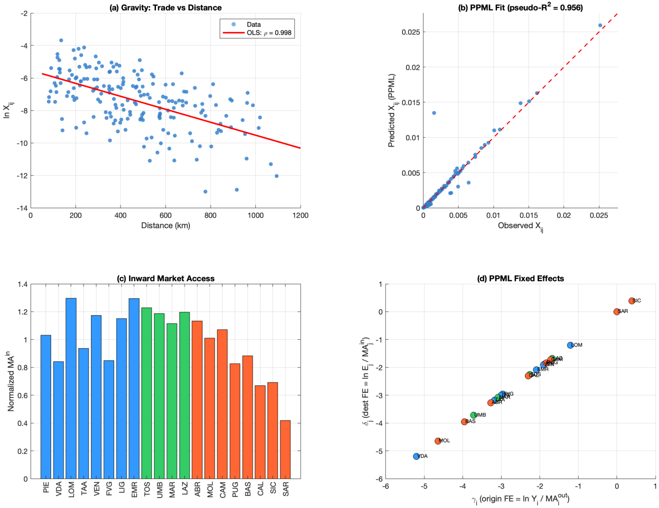
- **OLS** on $\ln X_{ij}$ with distance + origin/destination fixed effects
  — standard gravity; coefficient on distance recovers $(1-\sigma)\rho$
- **PPML** estimates in levels via iteratively reweighted least squares
  — robust to heteroskedasticity (Silva & Tenreyro, 2006); same coefficients

By-product: the origin/destination fixed effects encode market access.
$\hat{\gamma}_i = \ln(Y_i/\mathrm{MA}_i^{\mathrm{out}}), \quad \hat{\delta}_j = \ln(E_j/\mathrm{MA}_j^{\mathrm{in}})$.

With model-generated data, both OLS and PPML recover $\rho = 1$ exactly.

Gravity Estimation: Results

Figure 7: Gravity Estimation of Trade Costs (AA2025, Section 7.1)



Estimating α and β: The Identification Problem

The AA model implies supply and demand equations (eqs. 103–104):

$$\ln Y_i = \underbrace{(1+\alpha(\sigma-1))}_{\text{slope on } \ln L} \ln L_i \; - \; \ln \mathrm{MA}_i^{\mathrm{in}} \; - \; \underbrace{\ln C_i^S}_{\text{unobserved}} \qquad \text{(supply)}$$

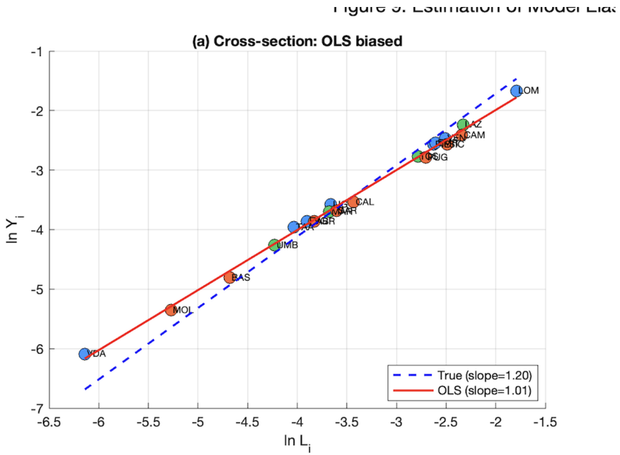
$$\ln Y_i = \underbrace{(1+\beta(\sigma-1))}_{\text{slope on } \ln L} \ln L_i \; + \; \ln \mathrm{MA}_i^{\mathrm{out}} \; + \; \underbrace{\ln C_i^D}_{\text{unobserved}} \qquad \text{(demand)}$$

Problem: C_i^S (amenities) and C_i^D (productivity) are unobserved and correlated with L_i and MA_i in equilibrium.
$\Rightarrow$ **Cross-sectional OLS is biased** (simultaneity).

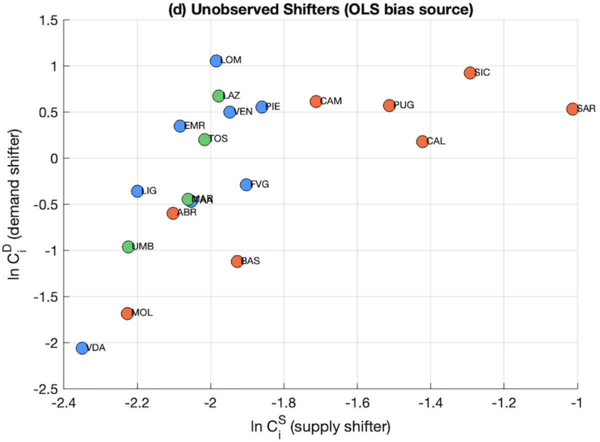
The code demonstrates this explicitly:

- Naive OLS gives wrong signs
- Controlling for the true C^S, C^D (structural verification) gives exact recovery
- Neither is feasible with real data — we need a different strategy

OLS gets it wrong



This is why


(d) Unobserved Shifters (OLS bias source)
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2.4
-2.2
-2
-1.8
-1.6
-1.4
-1.2
-1
ln C_i^D (demand shifter)
ln C_i^S (supply shifter)
LOM
SIC
LAZ
CAM
PUG
SAR
VEN
PIE
EMR
TOS
CAL
FVG
LIG
MAR
ABR
UMB
BAS
MOL
VDA

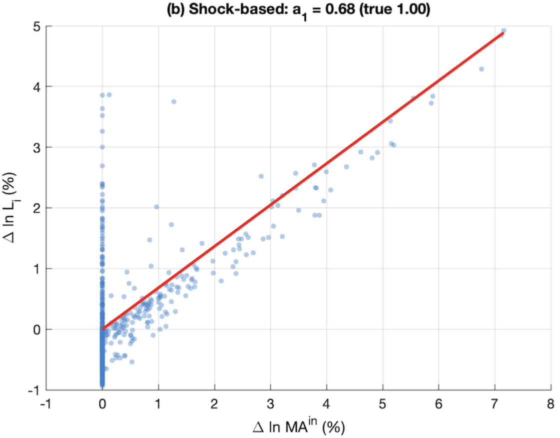
Shock-Based Identification (Reduced Form)

Strategy: use exogenous trade cost shocks and work in *first differences* to eliminate the fixed unobserved shifters C^S and C^D.

Policy-exposure instruments (Donaldson & Hornbeck, 2016)

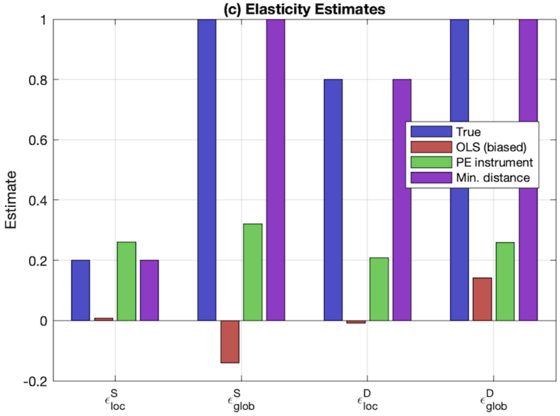
1. Generate 50 random **asymmetric** trade cost shocks
   — one-directional so $\Delta \ln \mathrm{MA}^{\mathrm{in}}$ and $\Delta \ln \mathrm{MA}^{\mathrm{out}}$ are not collinear
2. Compute $\Delta \ln \mathrm{MA}^{\mathrm{in,pol}}$ and $\Delta \ln \mathrm{MA}^{\mathrm{out,pol}}$ holding Y, E at baseline
   — a partial-equilibrium (PE) approximation to the true MA change
3. Regress $\Delta \ln L$ and $\Delta \ln Y$ on the two instruments (reduced form)
4. Recover structural elasticities: $\varepsilon^D_{\mathrm{local}} = 1 - b_1/a_1, \quad \varepsilon^S_{\mathrm{local}} = b_2/a_2 - 1$

Why this works



Estimates

Result: correct signs, but **substantial attenuation.**



Why the Attenuation Is Structural

The PE instrument computes $\Delta \ln \mathrm{MA}$ holding all other regions' wages, populations, and trade shares fixed. But in the true equilibrium, *even at first order*:

$$\text{shock to } \tau_{ij} \longrightarrow \text{wages change in } j \longrightarrow \text{competitiveness changes everywhere} \\ \longrightarrow \text{trade shares adjust} \longrightarrow \text{MA changes for all regions} \longrightarrow \text{migration} \longrightarrow \\ \cdots$$

The PE instrument captures only the **first link** in this chain. The GE multiplier is missing — systematically.

Key insight

This attenuation **does not vanish** with smaller shocks or more repetitions. It is a proportional bias, not a finite-sample or large-shock artefact. The PE instrument $\neq$ the true $\Delta \ln \mathrm{MA}$, even in the limit.

This is classic measurement error: $\Delta \ln \mathrm{MA}^{pol} = c \cdot \Delta \ln \mathrm{MA}^{true} + \text{noise}$ with $c < 1$.

Minimum Distance Estimation via Hat Algebra

The fix: embed the full GE model inside the estimation loop.

Minimum distance estimator

For each candidate (α, β):

1. Solve the baseline equilibrium $\longrightarrow$ trade shares π_{ij}, income Y_i
2. For each shock m: exact hat algebra $\longrightarrow$ $\Delta \ln L_i^{\text{model}}(\alpha, \beta)$, $\Delta \ln Y_i^{\text{model}}(\alpha, \beta)$
3. Evaluate the distance to "observed" changes:

$$\text{SSE}(\alpha, \beta) = \sum_{m,i} \left[\left(\Delta \ln L_{m,i}^{\text{obs}} - \Delta \ln L_{m,i}^{\text{model}} \right)^2 + \left(\Delta \ln Y_{m,i}^{\text{obs}} - \Delta \ln Y_{m,i}^{\text{model}} \right)^2 \right]$$

Find $(\hat{\alpha}, \hat{\beta}) = \arg\min \ \text{SSE}$ via Nelder–Mead.

Why it works: at the true parameters, hat algebra reproduces the observed changes exactly $\Rightarrow$ $\text{SSE} = 0$. No instruments, no attenuation.

Result: $\hat{\alpha} = 0.0500$ (true: 0.05), $\hat{\beta} = -0.2000$ (true: -0.20). **Exact recovery.**

Market Access Approach to Counterfactuals

With estimated elasticities, approximate counterfactual outcomes *without re-solving*:

$$\Delta \ln L_i \;\approx\; \underbrace{\frac{\varepsilon^S_{\text{global}}}{\varepsilon^S_{\text{local}} + \varepsilon^D_{\text{local}}}}_{\text{weight on demand side}} \;\Delta \ln \mathrm{MA}^{\text{in}}_i \;+\; \underbrace{\frac{\varepsilon^D_{\text{global}}}{\varepsilon^S_{\text{local}} + \varepsilon^D_{\text{local}}}}_{\text{weight on supply side}} \;\Delta \ln \mathrm{MA}^{\text{out}}_i$$

Symmetric case ($\tau_{ij} = \tau_{ji}$): $\mathrm{MA}^{\text{out}} \propto \mathrm{MA}^{\text{in}}$, so everything collapses to one regressor:

$$\Delta \ln Y_i \;\approx\; 1.4 \;\times\; \Delta \ln \mathrm{MA}^{\text{in}}_i$$

Applied to HSR corridor: the market access prediction tracks the full GE solution closely (correlation ≈ 1), giving policymakers a quick approximation tool.

Testing the Model

Test A — Predictive accuracy.
Apply a 25% North–South trade cost reduction, solve the full model ("observed data"), and check whether the market access approximation predicts the correct $\Delta \ln Y$ and $\Delta \ln L$.

Test B — Adão–Costinot–Donaldson primitive orthogonality.
Since the shock only changes τ (not $\bar{A}$ or $\bar{u}$), the recovered primitives should be *unchanged* between baseline and post-shock:

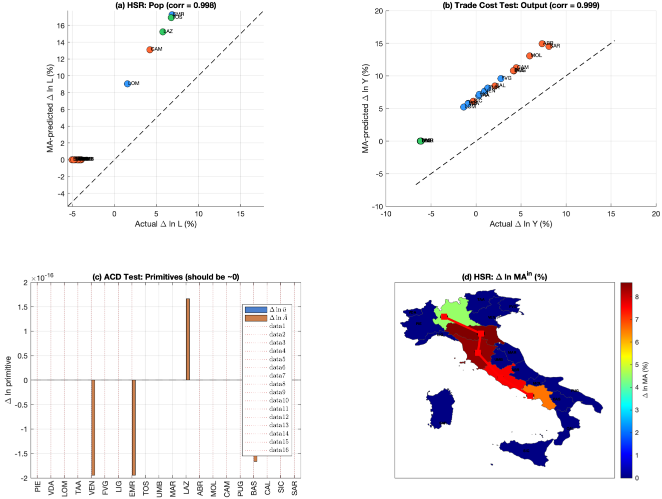
$$\Delta \ln \bar{A}_i \;\approx\; 0, \qquad \Delta \ln \bar{u}_i \;\approx\; 0 \qquad \forall\, i$$

If they are not, the model is misspecified.

Result: both tests pass. The maximum change in recovered primitives is $< 10^{-8}$ — the model is correctly specified.

Model Testing: Results

Figure 10: Model Testing -- Trade Cost Shock (AA2025, Section 7.3)



Summary

Hat algebra

- Not an approximation — exact reformulation
- Needs only observables + elasticities
- Avoids estimating $\bar{A}_i, \bar{u}_i$
- Linear version: fast but less accurate for large shocks

Estimation

- Gravity (PPML) recovers ρ
- Cross-sectional OLS is biased
- PE instruments: correct signs, structural attenuation
- Minimum distance via hat algebra: exact recovery of α, β

The practical message

In spatial GE models, **estimation and solution are inseparable.** Reduced-form instruments miss GE feedbacks and attenuate structurally. The hat algebra is not just a tool for counterfactuals — it is essential for estimation. A researcher who can solve the model can also estimate it, by embedding the solver inside the estimation loop (minimum distance / simulated method of moments).