Department of Economics
Ben-Gurion University
Yossi Yakhin

# Solving Linear Rational Expectations Models

## 1 Introduction

A large class of rational expectations models can be written as a system of linear stochastic first order difference equations. Linearity may be imposed by means of approximation, and a first order autoregressive structure can be achieved by appropriately renaming lagged variables of higher order. These notes review three methods of solving such models.

The first method is due to Blanchard and Kahn (1980). Their seminal paper analyzes a fairly restricted case, and the solution method was extended by King and Watson (1998), Uhlig (1999), and Klein (2000). This method requires the labeling of all endogenous variables as either state (i.e. pre-determined) variables or jump (i.e. non-predetermined) variables. Existence and uniqueness are then determined by comparing the number of explosive roots to the number of jump variables.

The second method is due to Christiano (2002). This method is based on the method of undetermined coefficients. As oppose to the first method it does not require the labeling of variables as predetermined or non-predetermined.

The third method is due to Sims (2002). As in Christiano's method there is no need to label variables as predetermined or non-predetermined. The method is different from the other two in that it explicitly exploits the properties of the expectational errors under rational expectations.

In what follows we review only simplified versions of these methods. This will allow us to focus on the main idea behind each of them without getting distracted by some complications that pose only technical difficulties. The procedure for the more general case us sketched at the end of these notes.

# 2 Method 1: Division to State and Jump Variables, Blanchard and Kahn (1980)

## 2.1 The Setup

Consider the following model:

$$\begin{aligned} \Gamma_0 E_t\left(x_{t+1}\right) &= \Gamma_1 x_t + \Psi z_t \qquad x_t \in \mathbb{R}^{p+m} \qquad z_t \in \mathbb{R}^k \qquad t \geq 0 \\ x_t &= \left[\begin{array}{cc} x_t^{s\prime} & x_t^{j\prime} \end{array}\right]' \qquad x_t^s \in \mathbb{R}^p \qquad x_t^j \in \mathbb{R}^m \\ E_t\left(x_{t+1}^s\right) &= x_{t+1}^s \qquad x_0^s \text{ given} \end{aligned} \tag{1}$$

where $x_t$ is a vector of endogenous variables, it is partitioned into a vector of state variables, $x_t^s$, and jump variables, $x_t^j$. Notice that the fact that $x_t^s$ is predetermined is reflected by the requirement $E_t\left(x_{t+1}^s\right) = x_{t+1}^s$. Finally, $z_t$ is an exogenous stationary process.

In what follows we restrict attention to stationary equilibria, specifically we assume that $E_t\left(x_{t+i}\right)$ is bounded for all $i \geq 0$.

### 2.1.1 Example

This example demonstrates how to cast a simple model into the form of (1). Consider the basic business cycle model augmented by exogenous productivity shocks, $A_t$. In this model agents maximize lifetime utility from consumption, and output is produced using capital alone. A solution to the model is characterized by:

$$\begin{aligned} U'\left(C_t\right) &= \beta E_t\left\{U'\left(C_{t+1}\right)\left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta\right)\right\} \\ C_t + K_{t+1} &= A_t K_t^\alpha + (1-\delta) K_t \\ A_t &= A_{t-1}^\rho \exp\left(\varepsilon_t\right) \qquad \varepsilon_t \overset{iid}{\sim} \left(0, \sigma^2\right) \qquad 0 < \rho < 1 \end{aligned}$$

The only difference from the deterministic version is that now productivity, $A_t$, is random. As a result agents must form expectations for the future path of $A_t$ before making consumption and investment decisions in each period. After log-linearization the model becomes:

$$
\begin{aligned}
\gamma \widetilde{C}_t &\cong \gamma E_t\left(\widetilde{C}_{t+1}\right) + \left[1-\beta\left(1-\delta\right)\right]\left[\left(1-\alpha\right)\widetilde{K}_{t+1} - \rho \widetilde{A}_t\right] \qquad \gamma \equiv -\frac{U''\left(C_{ss}\right)}{U'\left(C_{ss}\right)}C_{ss} \\
\widetilde{K}_{t+1} &\cong \frac{1-\beta\left(1-\delta\right)}{\alpha\beta}\widetilde{A}_t + \frac{1}{\beta}\widetilde{K}_t - \frac{1-\beta\left[1-\delta\left(1-\alpha\right)\right]}{\alpha\beta}\widetilde{C}_t \qquad (2) \\
\widetilde{A}_t &\cong \rho \widetilde{A}_{t-1} + \varepsilon_t \qquad \varepsilon_t \overset{iid}{\sim} \left(0, \sigma^2\right)
\end{aligned}
$$

where tilde variables denote log deviations from steady state. To cast the model into the form of (1) we need to define the vectors $x_t$ and $z_t$, and the coefficient matrices $\Gamma_0$, $\Gamma_1$, and $\Psi$, these are given by:

$$
\begin{aligned}
x_t &= \begin{bmatrix} \widetilde{K}_t & \widetilde{C}_t \end{bmatrix}' \qquad x_t^s \equiv \widetilde{K}_t \qquad x_t^j \equiv \widetilde{C}_t \qquad z_t \equiv \widetilde{A}_t \\
\Gamma_0 &\equiv \begin{bmatrix} \left[1-\beta\left(1-\delta\right)\right]\left(1-\alpha\right) & \gamma \\ 1 & 0 \end{bmatrix} \\
\Gamma_1 &\equiv \begin{bmatrix} 0 & \gamma \\ \frac{1}{\beta} & -\frac{1-\beta\left[1-\delta\left(1-\alpha\right)\right]}{\alpha\beta} \end{bmatrix} \\
\Psi &\equiv \begin{bmatrix} \left[1-\beta\left(1-\delta\right)\right]\rho \\ \frac{1-\beta\left(1-\delta\right)}{\alpha\beta} \end{bmatrix}
\end{aligned}
$$

## 2.2 Solution Method

We will make two simplifying assumptions. First, we assume that $\Gamma_0$ is invertible; therefore, we can write the model as:

$$
\begin{aligned}
E_t\left(x_{t+1}\right) &= A x_t + B z_t \qquad (3) \\
\text{where } A &= \Gamma_0^{-1}\Gamma_1 \\
\text{and } B &= \Gamma_0^{-1}\Psi
\end{aligned}
$$

Second, assume that the eigenvectors of $A$ are linearly independent. We can therefore use the eigen decomposition:

$$A = P\Lambda P^{-1}$$

where $\Lambda$ is a diagonal matrix with the eigenvalues of $A$ along its diagonal, and $P$ is a matrix of the corresponding eigenvectors. Note that we can always order the eigenvalues in $\Lambda$ (and arrange the columns of $P$ accordingly) in any way we wish. It proves convenient to order the elements of $\Lambda$ in exceeding order of absolute value. Specifically, let:

$$\Lambda = \begin{bmatrix} \underset{\overline{p}\times\overline{p}}{\Lambda_s} & \underset{\overline{p}\times\overline{m}}{0} \\ \underset{\overline{m}\times\overline{p}}{0} & \underset{\overline{m}\times\overline{m}}{\Lambda_e} \end{bmatrix}$$

where all diagonal elements of $\Lambda_s$ are smaller than 1 in absolute value, and all diagonal elements of $\Lambda_e$ are greater than 1 in absolute value.

Pre-multiply (3) by $P^{-1}$ to get:

$$\begin{aligned} E_t\left(w_{t+1}\right) &= \Lambda w_t + \overline{B} z_t \qquad (4) \\ \text{where } w_t &\equiv P^{-1} x_t \\ \overline{B} &\equiv P^{-1} B \end{aligned}$$

and partition $w_t$ and $\overline{B}$ to conform with the partition of $\Lambda$:

$$w_t = \begin{bmatrix} \underset{\overline{p}\times 1}{w_{1,t}} \\ \underset{\overline{m}\times 1}{w_{2,t}} \end{bmatrix} \quad , \quad \overline{B} = \begin{bmatrix} \underset{\overline{p}\times k}{\overline{B}_1} \\ \underset{\overline{m}\times k}{\overline{B}_2} \end{bmatrix}$$

Therefore, from (4) we get two sets of equations:

$$\begin{aligned} E_t\left(w_{1,t+1}\right) &= \Lambda_s w_{1,t} + \overline{B}_1 z_t \\ E_t\left(w_{2,t+1}\right) &= \Lambda_e w_{2,t} + \overline{B}_2 z_t \end{aligned}$$

The first equation is stable by construction because all diagonal elements in $\Lambda_s$ are less than 1 in absolute value and $z_t$ is stationary. The second equation, however, is explosive unless we choose the value of $w_{2,t=0}$ carefully. Taking expectations as of date $t$ and iterating forward gives:

$$w_{2,t} = \lim_{T\to\infty} \Lambda_e^{-T} E_t\left(w_{2,t+T}\right) - \textstyle\sum_{s=0}^{\infty} \Lambda_e^{-s-1}\overline{B}_2 E_t\left(z_{t+s}\right)$$

Since all diagonal elements of $\Lambda_e$ are greater than 1 in absolute value, and since $E_t\left(w_{2,t+T+1}\right)$ is bounded in any stationary equilibrium, the first expression converges to zero. Therefore the solution for $w_t$ is given by:

$$w_{2,t} = -\textstyle\sum_{s=0}^{\infty} \Lambda_e^{-s-1}\overline{B}_2 E_t\left(z_{t+s}\right) \tag{5}$$

We proceed by solving for $x_t^s$, $w_{1,t}$,and $x_t^j$ recursively. Recall that $x_t = Pw_t$, therefore:

$$x_{1,t} = P_{11}w_{1,t} + P_{12}w_{2,t} \tag{6a}$$

$$x_{2,t} = P_{21}w_{1,t} + P_{22}w_{2,t} \tag{6b}$$

where $x_{1,t}$ is a vector that consists of the first $\overline{p}$ elements of $x_t$, and $x_{2,t}$ consists of the remaining $\overline{m}$ elements. We will proceed by assuming that $p = \overline{p}$ and therefore $m = \overline{m}$, this suggests $x_{1,t} = x_t^s$ and $x_{2,t} = x_t^j$, and that $P_{11}$ is invertible. In this case if $x_t^s$ is known ($x_{t=0}^s$ is given) then the first equation solves for $w_{1,t}$:

$$w_{1,t} = P_{11}^{-1}x_t^s - P_{11}^{-1}P_{12}w_{2,t}$$

where $w_{2,t}$ is given by (5). Next, using the second equation we get $x_t^j$:

$$x_t^j = P_{21}w_{1,t} + P_{22}w_{2,t}$$

Finally, from (3) we get the value of the state variables in the next period, $x_{t+1}^s$:

$$x_{t+1}^s = A_{11}x_t^s + A_{12}x_t^j + B_1 z_t$$

where $A_{11}$ is the upper-left $\overline{p} \times \overline{p}$ block of $A$, $A_{12}$ is the upper-right $\overline{p} \times \overline{m}$ block of $A$, and $B_1$ is the upper $\overline{p} \times \overline{k}$ block of $B$. The remaining future values are found recursively.

After some substitutions, the recursive solution is summarized as follows:

1. Start from $t = 0$. $x^s_{t=0}$ is given.

2. Given $x^s_t$ and the realization of $z_t$, $x^j_t$ is given by:

$$x^j_t = P_{21}P_{11}^{-1}x^s_t + \left(P_{21}P_{11}^{-1}P_{12} - P_{22}\right)\sum_{s=0}^{\infty}\Lambda_e^{-s-1}\overline{B}_2 E_t\left(z_{t+s}\right)$$

3. The value of the state variables in the next period is given by:

$$x^s_{t+1} = A_{11}x^s_t + A_{12}x^j_t + B_1 z_t$$

4. Set $t = t + 1$, and go to step 2.

### 2.3 Existence and Uniqueness

In the derivation above we assumed that $p = \overline{p}$ and $m = \overline{m}$, that is the number of jump variables equals the number of explosive roots. For that case we found a unique equilibrium.

What happens if $m < \overline{m}$? In this case we can still solve for $w_{1,t}$ and $w_{2,t}$;[1] however, now (6b) imposes too many restrictions. Specifically, consider equation (6b) for $t = 0$:

$$x^2_{t=0} = P_{21}w_{1,t=0} + P_{22}w_{2,t=0}$$

Notice that the first $p - \overline{p}$ elements of $x^2_{t=0}$ are given since they are part of the state variables. Therefore, we do not have enough free variables, i.e. non-predetermined variables, to satisfy this equation. We conclude that if the number of jump variables is smaller than the number of explosive roots then solution does not exist.

What happens if $m > \overline{m}$? The problem now is that we cannot uniquely determine $w_{1,t}$. Consider (6a) for $t = 0$:

$$w_{1,t=0} = P_{11}^{-1}x^1_{t=0} - P_{11}^{-1}P_{12}w_{2,t=0}$$

Notice that the last $\overline{p} - p$ elements of $x^1_{t=0}$ are jump variables. Therefore this equation does not pin down uniquely $w_{1,t=0}$. In this case we are free to choose the initial value of any $\overline{p} - p$ jump variables. We conclude that if the number of jump variables is greater than the number of explosive roots then there is an infinity of solutions.

[1] Although solution for $w_{1t}$ is not unique.

# 3 Method 2: The Method of Undetermined Coefficients, Christiano (2002)

## 3.1 The Setup

Consider the following model:

$$\begin{aligned} \alpha_0 E_t\left(x_{t+1}\right) + \alpha_1 x_t + \alpha_2 x_{t-1} + \beta z_t &= 0 \qquad t \geq 0,\ x_{-1} \text{ given} \\ z_t &= R z_{t-1} + \varepsilon_t \qquad \varepsilon_t \overset{iid}{\sim} (0, \Sigma) \end{aligned} \tag{7}$$

where $x_t$ is an $n \times 1$ vector of endogenous variables that are determined at time $t$; and $z_t$ is a $k \times 1$ vector of exogenous stationary shocks. The matrices $\alpha_0$, $\alpha_1$, and $\alpha_2$ are $n \times n$ and $\beta$ is $n \times k$, and $R$ is $k \times k$. $E_t(\cdot)$ is the mathematical expectation conditional on information available at date $t$; this information includes current and past values of $x$ and $z$.

A notable difference from the method of Blanchard-Kahn is that we do not need to categorize variables as state or jump variables. This, however, does not come without cost since now we have to come up with initial values for the whole vector $x$ and not just to the state variables. This requirement imposes no difficulty if we start the system from steady state, but otherwise one must come up with an initial condition that is consistent with equilibrium.

A solution to the model is a feedback rule relating the current endogenous vector, $x_t$, to its past value, $x_{t-1}$, and the current exogenous shock, $z_t$. Since the model is linear, it is safe to guess that its solution is linear too, that is:

$$x_t = A x_{t-1} + B z_t \tag{8}$$

Restricting attention to stationary equilibria requires that *all* eigenvalues of $A$ must be less that 1 in absolute value. Although this may look odd at first glance, it is perfectly consistent with the Blanchard-Kahn approach since (7) requires initial values for all elements in $x_t$; therefore, we have no degrees of freedom to use for shutting down the explosive roots.[2]

[2]Notice that the elements of $x_t$ are variables that are determined at time $t$. This is different than the

The objective is therefore to find matrices $A$ and $B$ such that given $x_{-1}$ equation (8) is consistent with (7) and stationarity of $x_t$.

### 3.1.1 Example

This example demonstrates how to cast the stochastic business cycle model, as specified in (2), into the form of (7). To that end we need to define the vectors $x_t$ and $z_t$, and the coefficient matrices $\alpha_0$, $\alpha_1$, $\alpha_2$, and $\beta$, these are given by:

$$\begin{aligned}
x_t &= \begin{bmatrix} \widetilde{K}_{t+1} & \widetilde{C}_t \end{bmatrix}' \qquad z_t \equiv \widetilde{A}_t \\
\alpha_0 &\equiv \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix} \\
\alpha_1 &\equiv \begin{bmatrix} [1-\beta(1-\delta)](1-\alpha) & -\gamma \\ -1 & -\frac{1-\beta[1-\delta(1-\alpha)]}{\alpha\beta} \end{bmatrix} \\
\alpha_2 &\equiv \begin{bmatrix} 0 & 0 \\ \frac{1}{\beta} & 0 \end{bmatrix} \\
\beta &\equiv \begin{bmatrix} -[1-\beta(1-\delta)]\rho \\ \frac{1-\beta(1-\delta)}{\alpha\beta} \end{bmatrix}
\end{aligned}$$

## 3.2 Solution Method

Using (8), substitute for $x_{t+1}$ and $x_t$ in (7), then use $E_t(z_{t+1}) = Rz_t$, and get:

$$\left(\alpha_0 A^2 + \alpha_1 A + \alpha_2\right) x_{t-1} + \left[(\alpha_0 A + \alpha_1) B + \alpha_0 BR + \beta\right] z_t = 0 \tag{9}$$

Notice that this must be true for any realization of the exogenous shock, $z_t$, in particular it must hold for the case where $z_t = 0$ for all $t$. This observation suggests that the solution process can be divided into two steps, first shut down the exogenous shock and solve for the matrix $A$ and then, given $A$, solve for $B$.

Blanchard-Khan method that allows some variables to be pre-determined. For example, in Christiano's method the vector $x_t$ contains next period capital, $K_{t+1}$, while in the method of Blanchard and Khan $x_t$ contains current capital, $K_t$, that was determined in the previous period.

Specifically, notice that if $z_t = 0$ for all $t$, then the coefficient in front of $x_{t-1}$ must be zero. That is, $A$ is found by solving:

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0 \tag{10}$$

Now, since the first argument in (9) is zeroed out, it is clear that the coefficient in front of $z_t$ must be zero as well:

$$(\alpha_0 A + \alpha_1) B + \alpha_0 B R + \beta = 0 \tag{11}$$

Equation (10) is quadratic in the matrix $A$ and therefore may have more than one solution. An important question is how many solutions, if any, (10) has; and how many of those solutions satisfy the stationarity restriction that all eigenvalues are less than 1 in absolute value. These considerations will determine existence and uniqueness of the solution. Finally, Given $A$, equation (11) pins down $B$.

### 3.2.1 Solving for $A$

If $z_t = 0$ for at all times then there is no uncertainty and the model can be written as a deterministic $AR(1)$ process:

$$\Gamma_0 Y_{t+1} + \Gamma_1 Y_t = 0 \qquad t \geq 0 \tag{12}$$

where:

$$Y_t \equiv \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \quad , \quad \Gamma_0 \equiv \begin{bmatrix} \alpha_0 & 0_{n\times n} \\ 0_{n\times n} & I_n \end{bmatrix} \quad , \quad \Gamma_1 \equiv \begin{bmatrix} \alpha_1 & \alpha_2 \\ -I_n & 0_{n\times n} \end{bmatrix}$$

where $I_n$ denotes an $n \times n$ identity matrix, and $0_{n\times n}$ is an $n \times n$ matrix of zeros. Notice that the vector $Y_0$ is restricted by the $n$ initial conditions on $x_{-1}$.

In what follows we assume that $\Gamma_0$ is invertible.[3] In that case:

$$Y_{t+1} = -\Gamma_0^{-1} \Gamma_1 Y_t$$

[3]Notice that this is equivalent to assuming that $\alpha_0$ is invertible.

Let's also assume that the matrix $-\Gamma_0^{-1}\Gamma_1$ has $2n$ linearly independent eigenvectors. Therefore, by assumption we can apply the eigen decomposition and get $-\Gamma_0^{-1}\Gamma_1 = P\Lambda P^{-1}$, where $\Lambda$ is a diagonal matrix with the eigenvalues of $-\Gamma_0^{-1}\Gamma_1$ on its diagonal, and $P$ is a matrix of the corresponding eigenvectors.

Let $\overline{n}$ denote the number of stable eigenvalues. For convenience let $\Lambda_s$ be a diagonal matrix with the stable eigenvalues along its diagonal. Similarly, let $\Lambda_e$ be a diagonal matrix with the explosive eigenvalues. $\Lambda$ can now be organized with all stable eigenvalues in its first $\overline{n}$ diagonal elements:

$$\Lambda = \begin{bmatrix} \Lambda_{s,\overline{n}\times\overline{n}} & 0_{\overline{n}\times(2n-\overline{n})} \\ 0_{(2n-\overline{n})\times\overline{n}} & \Lambda_{e,(2n-\overline{n})\times(2n-\overline{n})} \end{bmatrix}$$

Using the eigen decomposition we get $P^{-1}Y_t = \Lambda P^{-1}Y_{t-1}$. Now define:

$$W_t \equiv P^{-1}Y_t \tag{13}$$

Therefore $W_t = \Lambda W_{t-1}$. Iterating backward gives:

$$W_t = \begin{bmatrix} \Lambda_s^t & 0 \\ 0 & \Lambda_e^t \end{bmatrix} W_0$$

By definition $W_t$ is a linear function of $x_t$ and $x_{t-1}$. Therefore stationarity of $x_t$ suggests that $W_t$ is stationary as well. The first $\overline{n}$ arguments in $W_t$ clearly satisfy this condition since $\Lambda_s^t \to 0$; however, the lower $2n - \overline{n}$ elements may be explosive. The only way the requirement for stationarity can be fulfilled is for the corresponding elements in $W_0$ to equal zero.

In what follows we assume that $\overline{n} = n$. By construction:

$$\begin{bmatrix} W_{1,0} \\ W_{2,0} \end{bmatrix} = \begin{bmatrix} P^{11} & P^{12} \\ P^{21} & P^{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_{-1} \end{bmatrix}$$

where $W_{1,0}$ is a vector of the first $n$ elements in $W_0$, $W_{2,0}$ is a vector of the last $n$ elements in $W_0$, and $P^{ij}$ is the $ij$ block in the matrix $P^{-1}$. For stationarity we need to impose $W_{2,0} = 0$, that is:

$$P^{21}x_0 + P^{22}x_{-1} = 0$$

and if $P^{21}$ is invertible, we get:

$$x_0 = -\left(P^{21}\right)^{-1} P^{22} x_{-1}$$

Which suggests:

$$A = -\left(P^{21}\right)^{-1} P^{22}$$

### 3.2.2 Solving for $B$

Equation (11) is linear in $B$. With the matrix $A$ in hand $B$ is given by:

$$vec\left(B\right) = -\left[I_k \otimes \left(\alpha_0 A + \alpha_1\right) + R' \otimes \alpha_0\right]^{-1} vec\left(\beta\right)$$

where the *vec* operator vectorizes the matrix $B$, that is it stacks the columns of $B$ into a vector, and $\otimes$ is a Kronecker product.[4] For the case $k = 1$ you can easily verify that this is a solution.

## 3.3 Existence and Uniqueness

In the derivation above we assumed that $\overline{n} = n$, that is the number of stable eigenvalues equals the number of elements in $x_t$. For that case we found a unique equilibrium.

Recall that by construction $W_0 = P^{-1}\left[\begin{array}{cc} x_0' & x_{-1}' \end{array}\right]'$, and that for stationarity we must zero out the last $2n - \overline{n}$ elements of $W_0$. We do that by choosing the elements of $x_0$.

What happens if $\overline{n} < n$? We need to zero out $2n - \overline{n} > n$ elements, but have only $n$ free variables. Therefore in this case equilibrium does not exist.

What happens if $\overline{n} > n$? We now need to zero out $2n - \overline{n} < n$ elements, and have $n$ free variables to do so. Therefore, there are many values of $x_0$ that are consistent with a stationary equilibrium; that is, equilibrium is not unique.

[4] If $A$ is $m \times n$ and $B$ is $p \times q$ then $A \otimes B$ is $mp \times nq$ block matrix:

$$A \otimes B = \left[\begin{array}{ccc} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{array}\right]$$

# 4 Method 3: Exploiting the Properties of Rational Expectational Errors, Sims (2002)

Consider the following model:

$$\Gamma_0 x_t = \Gamma_1 x_{t-1} + \Psi u_t + \Pi \eta_t \qquad t \geq 0 \qquad x_{-1} \text{ given} \tag{14}$$

where $x_t$ is an $n \times 1$ vector of endogenous variables, $\Gamma_0$ and $\Gamma_1$ are $n \times n$ coefficients matrices. $u_t$ is a $k \times 1$ vector of exogenous random disturbance, $\Psi$ is an $n \times k$ coefficients matrix. $\eta_t$ is an $r \times 1$ vector of expectational errors[5] satisfying $E_t\left(\eta_{t+1}\right) = 0$, and $\Pi$ is an $n \times r$ coefficients matrix. Note that under the assumption of rational expectations the $\eta_t$'s are endogenous. We will assume that the exogenous process, $u_t$, is a white noise, that is:

$$u_t \overset{iid}{\sim} (0, \Sigma) \tag{15}$$

This assumption is not very restrictive since given the structure of the model we can capture many different exogenous stationary processes simply by defining elements in $x_t$ properly. For example, if we wish to include the $AR(1)$ process: $z_t = \Phi z_{t-1} + u_t$, then $z_t$ should be included as an endogenous variable in $x_t$.

It should be noted that in this method the vector $x_t$ includes date $t$ expectation for the value of variables in $t+1$. These expectations are part of the endogenous variables in the model.

### 4.0.1 Example

This example demonstrates how to cast the stochastic business cycle model, as specified in (2), into the form of (14). To that end we need to define the vectors $x_t$, $u_t$, and $\eta_t$, and the

[5] The expectational errors are defined as the deviation of variables from their last period expected value, that is: $\eta_t = x_t - E_{t-1}(x_t)$.

coefficient matrices $\Gamma_0$, $\Gamma_1$, $\Psi$, and $\Pi$, these are given by:

$$
\begin{aligned}
x_t &= \begin{bmatrix} \widetilde{K}_{t+1} & \widetilde{C}_t & E_t\left(\widetilde{C}_{t+1}\right) & \widetilde{A}_t \end{bmatrix}' \qquad u_t \equiv \varepsilon_t \qquad \eta_t \equiv \widetilde{C}_t - E_{t-1}\left(\widetilde{C}_t\right) \\
\Gamma_0 &\equiv \begin{bmatrix} [1-\beta(1-\delta)](1-\alpha) & -\gamma & \gamma & -[1-\beta(1-\delta)]\rho \\ 1 & \frac{1-\beta[1-\delta(1-\alpha)]}{\alpha\beta} & 0 & -\frac{1-\beta(1-\delta)}{\alpha\beta} \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
\Gamma_1 &\equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \Psi \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \Pi \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

Notice that this system includes an additional equation relative to (2) that defines the expectational error, $\eta_t$.

## 4.1 Solution Method

As in the previous methods we will assume for simplicity that $\Gamma_0$ is invertible. We can therefore write the model as:

$$
\begin{aligned}
x_t &= Ax_{t-1} + \Gamma_0^{-1}(\Psi u_t + \Pi\eta_t) \qquad t \geq 1 \qquad x_0 \text{ given} \\
\text{where } A &\equiv \Gamma_0^{-1}\Gamma_1
\end{aligned}
$$

Again, assuming that all eigenvectors of $A$ are linearly independent we can apply the eigen decomposition: $A = P\Lambda P^{-1}$. Pre-multiply by $P^{-1}$, and define $w_t \equiv P^{-1}x_t$:

$$
\begin{aligned}
w_t &= \Lambda w_{t-1} + Q(\Psi u_t + \Pi\eta_t) \qquad t \geq 1 \\
\text{where } Q &\equiv P^{-1}\Gamma_0^{-1}
\end{aligned}
$$

As before, we organize the eigenvalues in exceeding order, and partition $\Lambda$ such that:

$$
\Lambda = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_e \end{bmatrix}
$$

where $\Lambda_s$ is diagonal with all elements smaller than 1 in absolute value, and $\Lambda_e$ is diagonal with all elements greater than 1 in absolute value.

The system as a whole can be partitioned into two sets of equations:

$$\begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_e \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} (\Psi u_t + \Pi \eta_t) \qquad t \geq 1$$

The upper block is stable and the lower one is explosive. We start by solving for the explosive block.

### 4.1.1 The Explosive Block

Rearranging the explosive block we get:

$$w_{2,t} = \Lambda_e^{-1} w_{2,t+1} - \Lambda_e^{-1} Q_2 \left( \Psi u_{t+1} + \Pi \eta_{t+1} \right) \tag{16}$$

Iterating forward:

$$w_{2,t} = \lim_{T \to \infty} \Lambda_e^{-T} w_{2,t+T} - \sum_{s=1}^{\infty} \Lambda_e^{-s} Q_2 \left( \Psi u_{t+s} + \Pi \eta_{t+s} \right)$$

Taking expectations given date $t$ information:

$$w_{2,t} = \lim_{T \to \infty} \Lambda_e^{-T} E_t \left( w_{2,t+T} \right) - \sum_{s=1}^{\infty} \Lambda_e^{-s} Q_2 E_t \left( \Psi u_{t+s} + \Pi \eta_{t+s} \right)$$

In a stationary equilibrium $E_t \left( w_{2,t+T} \right)$ is bounded, and since all elements in $\Lambda_e$ are greater than 1 in absolute value, $\lim_{T \to \infty} \Lambda_e^{-T} = 0$. Therefore, $\lim_{T \to \infty} \Lambda_e^{-T} E_t \left( w_{2,t+T} \right) = 0$. In addition, $u_t$ is a white noise and under rational expectations the $\eta$'s must have conditional mean of zero; therefore, the second expression is zeroed out as well. This gives:[6]

$$w_{2,t} = 0 \tag{17}$$

### 4.1.2 Solving for Expectational Errors

Before solving for $w_{1,t}$ we want to get rid of the expectational errors. From (16) and (17) we get:

$$Q_2 \left( \Psi u_t + \Pi \eta_t \right) = 0 \tag{18}$$

[6] Notice that the condition $w_{2,t=0} = \begin{bmatrix} P^{21} & P^{22} \end{bmatrix} x_{t=0} = 0$ requires the linear combinations of $x_0$ that are associated with the explosive roots to be zero. If we labeled variables as "jump" and "state" this restriction could provide the conditions for existence and uniqueness of a stationary equilibrium in a similar manner to the method of Blanchard-Kahn.

which suggests that the expectational errors move only with new exogenous information, as should be the case under rational expectations.

We will continue by assuming that $Q_2\Pi$ is invertible. This assumption allows us to uniquely determine $\eta_t$:

$$\eta_t = -\left(Q_2\Pi\right)^{-1} Q_2\Psi u_t$$

#### 4.1.3 The Stable Block

The stable block is given by:

$$w_{1,t} = \Lambda_s w_{1,t-1} + Q_1\left(\Psi u_t + \Pi\eta_t\right)$$

substituting for $\eta_t$ gives:

$$w_{1,t} = \Lambda_s w_{1,t-1} + Q_1\left(\Psi - \Pi\left(Q_2\Pi\right)^{-1} Q_2\Psi\right) u_t \tag{19}$$

#### 4.1.4 Solving for $x_t$

Equations (17) and (19) solve for $w_t$:

$$\begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \Lambda_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} Q_1\left(\Psi - \Pi\left(Q_2\Pi\right)^{-1} Q_2\Psi\right) \\ 0 \end{bmatrix} u_t$$

By using $x_t = Pw_t$ we can easily back up the solution for $x_t$. Pre-multiply by $P$ and get:

$$x_t = P \begin{bmatrix} \Lambda_s & 0 \\ 0 & 0 \end{bmatrix} P^{-1} x_{t-1} + P \begin{bmatrix} Q_1\left(\Psi - \Pi\left(Q_2\Pi\right)^{-1} Q_2\Psi\right) \\ 0 \end{bmatrix} u_t$$

### 4.2 Existence and Uniqueness

The condition for existence is derived from (18), which is restated here for convenience:

$$Q_2\left(\Psi u_t + \Pi\eta_t\right) = 0$$

Given the exogenous shocks, $u_t$, this equation determines the expectational errors, $\eta_t$. Existence problems arise if $\eta_t$ cannot adjust to offset $u_t$. This may happen if equation (18) puts too many restrictions for the expectational errors to satisfy. A necessary and sufficient condition for a solution to exist is that the column space of $Q_2\Psi$ be contained in that of $Q_2\Pi$.

Multiple solutions may exist when (18) puts too few restrictions; in that case there are infinitely many combinations of expectational errors that satisfy these restrictions. Therefore, equation (18) may only pin down $Q_2\Pi\eta_t$, not $\eta_t$. Now recall that the the stable block involves a different linear combinations of the expectational errors, $Q_1\Pi\eta_t$. It may be the case that knowledge of $Q_2\Pi\eta_t$ is not enough to pin down $Q_1\Pi\eta_t$, if that is the case the system has multiple solutions. A necessary and sufficient condition for a solution to be unique is that the row space of $Q_1\Pi$ be contained in that of $Q_2\Pi$. That is, a solution is unique if there exists a matrix $\Phi$ such that: $Q_1\Pi = \Phi Q_2\Pi$.

# 5 Non-invertible $\Gamma_0$ and Non-diagonalizable Matrices

Throughout these notes we assumed that $\Gamma_0$ is invertible, and that the product $\Gamma_0^{-1}\Gamma_1$ is diagonalizable. This allowed us to transform the model into a set of scalar equations that are easier to handle. In what follows we sketch the solution procedure for the case where these assumptions are not satisfied.

For simplicity, consider the deterministic model:

$$\begin{aligned} \Gamma_0 x_{t+1} &= \Gamma_1 x_t \qquad t \geq 0 \qquad x_0 \text{ given} \\ x_t &\in \mathbb{R}^n \end{aligned}$$

We now allow the matrix $\Gamma_0$ to be singular. Since we cannot diagonalize the system, we will triangulize it instead. To that end, we need to use the $QZ$ decomposition.

**Definition 1 (QZ decomposition)** [7] *For square matrices $\Gamma_0$ and $\Gamma_1$ there exist unitary*

[7] Strictly speaking, for the $QZ$ decomposition to always exist the definition must allow for complex roots. Here we abstract from that possibility.

*matrices $Q$ and $Z$, i.e. $QQ' = ZZ' = I$, and upper triangular matrices $\Omega$ and $\Lambda$ such that:*

$$\begin{aligned} \Gamma_0 &= Q'\Omega Z' \\ \Gamma_1 &= Q'\Lambda Z' \end{aligned} \tag{20}$$

The collection of values of the ratios of diagonal elements of $\Lambda$ and $\Omega$, $\left\{\frac{\lambda_{ii}}{\omega_{ii}}\right\}$, is the set of generalized eigenvalues of $\Gamma_0$ and $\Gamma_1$.[8] The set of generalized eigenvalues is usually unique if we include $\infty$ as a possible value (when $\Gamma_0$ is singular some of the $\omega_{ii}$'s might be zero). We can always choose the matrices $Q$, $Z$, $\Omega$, and $\Lambda$ in a way that the generalized eigenvalues are organized in exceeding order of their absolute value.

Using (20), pre-multiply the model by $Q$ and get:

$$\Omega Z' x_{t+1} = \Lambda Z' x_t$$

Define $w_t \equiv Z' x_t$, and we get a *triangular* system in $w_t$:

$$\Omega w_{t+1} = \Lambda w_t \tag{21}$$

Since all explosive roots, i.e. $\left|\frac{\lambda_{ii}}{\omega_{ii}}\right| > 1$, are concentrated at the bottom we can partition the system into a stable block and an explosive one:

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix}$$

As usual, we start from the explosive block.

## 5.1 The Explosive Block

Notice that by construction $\Lambda_{22}$ is invertible, pre-multiply the explosive block by $\Lambda_{22}^{-1}$ and iterate forward to get:

$$w_{2,t} = \lim_{T\to\infty} \left(\Lambda_{22}^{-1}\Omega_{22}\right)^T w_{2,t+T}$$

[8] The generalized eigenvalue problem has the form: $Ax = \lambda Bp$. Where $A$ and $B$ are $n \times n$ matrices, $p \neq 0$ is an eigenvector, and $\lambda$ is an eigenvalue. In the standard eigenvalue problem $B = I$.

Notice that since $\Omega$ and $\Lambda$ are upper triangular the diagonal elements of $\Lambda_{22}^{-1}\Omega_{22}$ are $\frac{\omega_{ii}}{\lambda_{ii}}$, i.e. the inverse of the generalized eigenvalues. The last equation in this system is a scalar equation, and since $\lim_{T\to\infty}\left(\frac{\omega_{nn}}{\lambda_{nn}}\right)^T = 0$ together with the requirement for stationarity we get that the last component in $w_{2,t}$ is zeroed out. Using this result and following the same steps we conclude that the second to last element in $w_{2,t}$ is also zero, and so on. We can therefore conclude that in any stationary equilibrium:

$$w_{2,t} = 0$$

## 5.2 The Stable Block

By construction $\Omega_{11}$ is invertible. Pre-multiplying the stable block by $\Omega_{11}^{-1}$, using $w_{2,t} = 0$, and iterating the stable block backward, we get:

$$w_{1,t} = \left(\Omega_{11}^{-1}\Lambda_{11}\right)^t w_{1,t=0}$$

and $w_{1,0}$ is known from the initial conditions on $x_0$.

## 5.3 Solving for $x_t$

Given the solution for $w_t$ we can easily back up the solution for $x_t$. Recall that $w_t = Z'x_t$ and that $Z$ is unitary, therefore:

$$x_t = Zw_t$$

# 6 Matlab Programs

These notes described three methods for solving linear rational expectations models. In practice these models are solved numerically on the computer. Several Matlab programs are available freely on the web as many authors complement their papers with a program that implement their suggested solution method.

As an end-user of these programs you don't really need to have any understanding of the solution method. What you are required is to transform your model into the appropriate form according to your package of choice and provide the program with the coefficient matrices of the model. The computer will do the rest. Here are some links to such packages:

- Matlab code for the method of Klein (2000) (an extension to Blanchard-Kahn):

  http://economics.uwo.ca/faculty/klein/personal/?codes.htm

- Matlab code for the method of Uhlig (1999) (an extension to Blanchard-Kahn):

  http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm

  The code provides a complete toolkit for simulations. Several add-ons from different authors are also available.

- Matlab code for the method of Christiano (2002) (undetermined coefficient method):

  http://www.faculty.econ.northwestern.edu/faculty/christiano/research/Solve/main.htm

- Matlab code for the method of Sims (2002) (expectational errors method):

  http://sims.princeton.edu/yftp/gensys/

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